IRREDUCIBLE CHARACTERS OF $\mathrm{GSp}(4, \mathbb{F}_q)$

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ABSTRACT. Admissible non-supercuspidal representations of $\mathrm{GSp}(4, F)$, where F is a local field of characteristic zero with an odd-ordered residue field $\mathbb{F}_q$, have finite dimensional spaces of fixed vectors under the action of principal congruence subgroups. We can say precisely what these dimensions are for nearly all local fields and principal congruence subgroups of level $\mathfrak{p}$ by understanding the non-cuspidal representation theory of the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$. The conjugacy classes and the list of irreducible characters of this group are given. Genericity and cuspidality of the irreducible characters are also determined.

CONTENTS

1. INTRODUCTION

A useful approach to finding dimensions of spaces of Siegel cusp forms is to investigate the representation theory of $\mathrm{GSp}(2n)$. Results in this group's representation theory can be translated to results on spaces of cusp forms and vice-versa. Cuspidal representations are the building blocks for the representation theory of

certain groups in a way analogous to the construction of Eisenstein series from cusp forms. More precisely, a cuspidal Siegel modular form $f \in S_k(\Gamma(N))$ of degree n gives rise to a cuspidal automorphic representation (π, V) of $\mathrm{GSp}(2n, \mathbb{A})$ and vice-versa. These cuspidal automorphic representations π can be written in terms of local components π_ν, where ν is a place of $\mathbb{Q}$. The local components of the automorphic representation in turn give rise to local components of the cusp form. An example of using this correspondence is the following, where one can find the dimensions of spaces where some of these local components live. The dimensions tell us essentially how many choices we have for the local factors of the representation and therefore the number of choices of local vectors.

Let F be a non-archimedean local field of characteristic zero with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$ such that $\mathfrak{o}/\mathfrak{p}$ is isomorphic to $\mathbb{F}_q$, the finite field of order $q = p^n$ for p an odd prime. Then $\mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \cong \mathrm{GSp}(4, \mathbb{F}_q)$. Consider the group $\mathrm{GSp}(4, F)$ and hence Siegel modular forms of degree 2. Define the principal congruence subgroup of level $\mathfrak{p}^n$, denoted by $\Gamma(\mathfrak{p}^n)$, by

$$\Gamma(\mathfrak{p}^n) = \{g \in \mathrm{GSp}(4, \mathfrak{o}) \,:\, g \equiv I \,(\mathrm{mod}\,\mathfrak{p}^n)\}$$

For the maximal compact subgroup $K = \mathrm{GSp}(4, \mathfrak{o})$ and an admissible representation (π, V) of $\mathrm{GSp}(4, F)$, K acts on the space $V^{\Gamma(\mathfrak{p})}$ of vectors in V fixed by the action of the congruence subgroup $\Gamma(\mathfrak{p})$. This space is finite dimensional by the admissibility of the representation. By definition, $\Gamma(\mathfrak{p})$ acts trivially on this space and so we have a more interesting action of the group $K/\Gamma(\mathfrak{p}) \cong \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \cong \mathrm{GSp}(4, \mathbb{F}_q)$. The dimension of $V^{\Gamma(\mathfrak{p})}$ can then be determined by looking at the finite group analogue of π.

Our first step in determining these analogues is to find the list of conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$. This list is used to compute the classes of the Borel, the Siegel parabolic, and the Klingen parabolic subgroups. The main tool we use for determining the conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ is a paper of Wall [19], which was also used to determine the conjugacy classes of the symplectic group $\mathrm{Sp}(4, \mathbb{F}_q)$ in Srinivasan's paper [18]. An isomorphism between $\mathrm{GSp}(4, \mathbb{F}_q)$ modulo its center and a special orthogonal group is used to solve the conjugacy class problem for $\mathrm{GSp}(4, \mathbb{F}_q)$. We then determine how the conjugacy classes split in the Borel, the Siegel parabolic, and the Klingen parabolic subgroups.

All of the irreducible characters of the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$ and their cuspidality and genericity are determined. Cuspidality is determined by defining cuspidal representations on the Borel, the Siegel parabolic, and the Klingen parabolic subgroup and then inducing. The irreducible non-cuspidal representations are precisely the irreducible constituents of these induced representation. Criteria are determined for these induced characters to be irreducible. If an induced character is reducible, then the constituents are found.

2. Basic definitions and notations

Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements, with p an odd prime.

Definition 2.1. The group $G = \mathrm{GSp}(4, \mathbb{F}_q)$ is defined as

$$\mathrm{GSp}(4, \mathbb{F}_q) := \{g \in \mathrm{GL}(4, \mathbb{F}_q) : {}^t g J g = \lambda J\}, \text{ where } J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_q^\times$, which will be denoted by $\lambda(g)$ and called the *multiplier* of g. The set of all $g \in \mathrm{GSp}(4, \mathbb{F}_q)$ such that $\lambda(g) = 1$ is the subgroup $\mathrm{Sp}(4, \mathbb{F}_q)$.

Let κ be a generator of $\mathbb{F}_{q^4}^\times$ and let $\zeta = \kappa^{q^2-1}$, $\theta = \kappa^{q^2+1}$, $\eta = \theta^{q-1}$, and $\gamma = \theta^{q+1}$. The element η is the generator of the set of elements in $\mathbb{F}_{q^2}$ whose norm over $\mathbb{F}_q$ is 1 and ζ is the generator of the set of elements in $\mathbb{F}_{q^4}$ whose norm over $\mathbb{F}_{q^2}$ is 1. Define the sets

$R_1 = \{1, ..., \frac{1}{4}(q^2 - 1)\}$,

R_2 is a set of $\frac{1}{4}(q-1)^2$ distinct positive integers i such that θ^i, θ^{-i}, θ^{qi}, and θ^{-qi} are all distinct,

$T_1 = \{1, ..., \frac{1}{2}(q - 3)\}$,

$T_2 = \{1, ..., \frac{1}{2}(q - 1)\}$,

$T_3 = \{1, ..., q - 1\}$.

Note that for any $g \in G$, we can uniquely write g as

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \lambda(g) & \\ & & & \lambda(g) \end{pmatrix} \cdot g',$$

where $g' \in \mathrm{Sp}(4, \mathbb{F}_q)$.

The order of $\mathrm{Sp}(4, \mathbb{F}_q)$, as computed by Wall, is $q^4(q^4 - 1)(q^2 - 1)$. So the order of $\mathrm{GSp}(4, \mathbb{F}_q)$ is $q^4(q^4 - 1)(q^2 - 1)(q - 1)$.

3. CONJUGACY CLASSES

The conjugacy classes of the unitary, symplectic and orthogonal groups can be determined using the results of Wall [19]. Srinivasan, in [18], used Wall's results to explicitly determine the conjugacy classes of $\mathrm{Sp}(4, \mathbb{F}_q)$.

Wall's results cannot be directly used to determine the conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ but they can be used to find the classes of $\mathrm{SO}(5, \mathbb{F}_q)$. This is particularly useful because $\mathrm{SO}(5, \mathbb{F}_q)$ is isomorphic to $\mathrm{PGSp}(4, \mathbb{F}_q) := \mathrm{GSp}(4, \mathbb{F}_q)/Z$, where Z is the center of $\mathrm{GSp}(4, \mathbb{F}_q)$. These classes are then used to determine the classes of $\mathrm{GSp}(4, \mathbb{F}_q)$. Explicitly, the list of conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ is given in the following table.

Table 1: Conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$

Notation	Class representative	Number of classes	Order of centralizer
$A_1(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q - 1$	$\#\mathrm{GSp}(4, \mathbb{F}_q)$
$A_2(k)$, $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & \gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q - 1$	$q^4(q^2 - 1)(q - 1)$

Table 1 – Continued

Notation	Class representative	Number of classes	Order of centralizer
$A_{31}(k),$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q-1)^2$
$A_{32}(k),$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \gamma^{k+1} \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$2q^3(q^2-1)$
$A_5(k),$ $k \in T_3$	$\begin{pmatrix} \gamma^k & \gamma^k & -\gamma^k & \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & -\gamma^k \\ & & & \gamma^k \end{pmatrix}$	$q-1$	$q^2(q-1)$
$B_{11}(k),$ $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q^2(q^2-1)^2(q-1)$
$B_{12}(k),$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+1} & \gamma^k & & \\ & & & \\ & & \gamma^{k+1} & \gamma^k \\ & & & \end{pmatrix}$	$\dfrac{q-1}{2}$	$q^2(q^4-1)(q-1)$
$B_{21}(k),$ $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & -\gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q^2-1)(q-1)^2$
$B_{22}(k),$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+1} & \gamma^k & & \\ & & & \\ & & -\gamma^{k+1} & -\gamma^k \\ & & & \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q^2-1)^2$
$B_3(k),$ $k \in T_3$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$q-1$	$q^2(q^2-1)(q-1)$

Table 1 – Continued

Notation	Class representative	Number of classes	Order of centralizer
$B_{41}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^k & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2},$	
$B_{42}(k)$, $k \in T_2$	$\begin{pmatrix} -\gamma^k & & & \gamma^k \\ & \gamma^k & -\gamma^{k+1} & \\ & & \gamma^k & \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2},$	$2q^2(q-1)$
$B_{43}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^{k+1} & \gamma^k & & -\frac{a}{2}\gamma^k \\ & -\frac{a}{2}\gamma^{k+1} & b\gamma^{k+1} & \\ & & & \gamma^k \\ & & \gamma^{k+1} & \end{pmatrix}$	$\dfrac{q-1}{2},$	
$B_{44}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^{k+1} & \gamma^k & c\gamma^k & \\ & & & 2\gamma^{k+1} \\ & & & \gamma^k \\ & \gamma^{k+1} & & \end{pmatrix}$	$\dfrac{q-1}{2}$	
$B_{51}(k)$, $k \in T_2$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & -\gamma^k & -\gamma^k \\ & & & -\gamma^k \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q-1)^2$
$B_{52}(k)$, $k \in T_2$	$\begin{pmatrix} \frac{1}{2}\gamma^k & 2\gamma^{k+1} & \frac{1}{2}\gamma^k & \\ \frac{1}{2}\gamma^k & & & \\ & & & -2\gamma^{k+1} \\ & & -\frac{1}{2}\gamma^k & \end{pmatrix}$	$\dfrac{q-1}{2}$	$q(q^2-1)$
$C_1(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$C_{21}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^k & & & \\ & -\gamma^k & & \\ & & \gamma^{k+i} & \\ & & & -\gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{4}$	$(q-1)^3$
$C_{22}(i,k)$, $i \in T_1$ $k \in T_2$	$\begin{pmatrix} \gamma^{k+1} & \gamma^k & & \\ & & & -\gamma^{k+i} \\ & & -\gamma^{k+i+1} & \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{4}$	$(q^2-1)(q-1)$

Table 1 – Continued

Notation	Class representative	Number of classes	Order of centralizer
$C_3(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & -\gamma^k & & \\ & \gamma^k & & \\ & & \gamma^{k+i} & \gamma^{k+i} \\ & & & \gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$C_4(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^{k+i} & & \\ & & \gamma^{k-i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q-1)^2$
$C_5(i,k)$, $i \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^{k+i} & & & \\ & \gamma^k & & \\ & & \gamma^k & \\ & & & \gamma^{k-i} \end{pmatrix}$	$\dfrac{(q-1)(q-3)}{2}$	$q(q^2-1)(q-1)^2$
$C_6(i,j,k)$, $i,j \in T_1$ $i < j$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^{k+i} & & \\ & & \gamma^{k+j} & \\ & & & \gamma^{k+i+j} \end{pmatrix}$	$\dfrac{(q-1)(q-3)(q-5)}{8}$	$(q-1)^3$
$D_1(i,k)$, $i \in R_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k\theta^i & & \\ & & \gamma^k\theta^{qi} & \\ & & & \gamma^{k+i} \end{pmatrix}$	$\dfrac{(q-1)^3}{4}$	$(q^2-1)(q-1)$
$D_2(i,k)$, $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & \gamma^k\theta^i & \\ & & & \gamma^k\theta^{qi} \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)^2$
$D_{31}(i,k)$, $i \in T_2$ $k \in T_2$	$\begin{pmatrix} -\gamma^k\theta^i & & & \\ & \gamma^k\theta^{qi} & & \\ & & -\gamma^k\theta^{qi} & \\ & & & \gamma^k\theta^i \end{pmatrix}$	$\dfrac{(q-1)^2}{4}$	$(q^2-1)(q-1)$
$D_{32}(i,k)$, $i \in T_2$ $k \in T_2$	$D_{32}(i,k)$	$\dfrac{(q-1)^2}{4}$	$(q^2-1)(q+1)$

Table 1 – Continued

Notation	Class representative	Number of classes	Order of centralizer
$D_4(i,j,k),$ $i \in T_2$ $j \in T_1$ $k \in T_3$	$\begin{pmatrix} \gamma^k \theta^{qi} & & & \\ & \gamma^k \theta^i & & \\ & & \gamma^{k+j} \theta^{qi} & \\ & & & \gamma^{k+j} \theta^i \end{pmatrix}$	$\dfrac{(q-1)^2(q-3)}{4}$	$(q^2-1)(q-1)$
$D_5(i,k),$ $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k \theta^i & -\gamma^k \theta^i & & \\ & \gamma^k \theta^i & & \\ & & \gamma^k \theta^{qi} & \gamma^k \theta^{qi} \\ & & & \gamma^k \theta^{qi} \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)$
$D_6(i,k),$ $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & \\ & \gamma^k \eta^i & & \\ & & \gamma^k \eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)^2$
$D_7(i,j,k),$ $i,j \in T_2$ $i < j$ $k \in T_3$	$D_7(i,j,k)$	$\dfrac{(q-1)^2(q-3)}{8}$	$(q^2-1)(q+1)$
$D_8(i,k),$ $i \in T_2$ $k \in T_3$	$\begin{pmatrix} \gamma^k & & & -\gamma^k \\ & \gamma^k \eta^i & & \\ & & \gamma^k \eta^{-i} & \\ & & & \gamma^k \end{pmatrix}$	$\dfrac{(q-1)^2}{2}$	$q(q^2-1)$
$D_9(i,k),$ $i \in R_1$ $k \in T_3$	$D_9(i,k)$	$\dfrac{(q^2-1)(q-1)}{4}$	$(q^2+1)(q-1)$

where $a,b \in \mathbb{F}_q^\times$ s.t. $-a^2+b^2\gamma$ is a square and $c = \gamma+1$. There are $(q^2+2q+4)(q-1)$ conjugacy classes. Explicit forms of the classes $D_{32}(i,k)$, $D_7(i,j,k)$, and $D_9(i,k)$ are omitted since they have a more complicated form and are not classes of the Borel, the Siegel parabolic, or the Klingen parabolic subgroup. The reader may notice that some of the class representatives are in a form with entries not in $\mathbb{F}_q$, but in $\mathbb{F}_{q^2}$. This is done to more immediately indicate the eigenvalues.

4. Borel, Siegel parabolic, and Klingen parabolic subgroups

The conjugacy classes of the Borel, the Siegel parabolic, and the Klingen parabolic subgroup can now be easily computed using Table 1. One does this by determining which conjugacy classes have a non-empty intersection with the subgroup, determining how each class splits, and computing the order of the centralizer of the class in the subgroup.

4.1. **Borel.** The Borel subgroup B of $\mathrm{GSp}(4, \mathbb{F}_q)$ is the set of all of the upper triangular matrices,

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{F}_q) \right\}.$$

Every element $g \in B$ can be written uniquely in the form

$$g = \begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

with $a, b, c \in \mathbb{F}_q^{\times}$ and $x, \kappa, \lambda, \mu \in \mathbb{F}_q$. The order of B is therefore $q^4(q-1)^3$. The multiplier of the matrix g given above is $\lambda(g) = c$. The subgroup of B of elements which have 1 on every entry on the main diagonal is denoted by $N_{\mathrm{GSp}(4)}$.

4.2. **Siegel.** The Siegel parabolic subgroup P of $\mathrm{GSp}(4, \mathbb{F}_q)$ is the set of all block upper triangular matrices. That is,

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{F}_q) \right\}.$$

Every element $p \in P$ can be written uniquely in the form

$$p = \begin{pmatrix} a & b & & \\ c & d & & \\ & & \lambda a/\Delta & -\lambda b/\Delta \\ & & -\lambda c/\Delta & \lambda d/\Delta \end{pmatrix} \cdot \begin{pmatrix} 1 & & \mu & \kappa \\ & 1 & x & \mu \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

with $\Delta = ad - bc \in \mathbb{F}_q^{\times}$, $\lambda \in \mathbb{F}_q^{\times}$ and $x, \kappa, \mu \in \mathbb{F}_q$. The order of P is therefore $q^4(q^2 - 1)(q - 1)^2$. The multiplier of p is $\lambda(p) = \lambda$. We also define

$$A' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

for any $A \in \mathrm{GL}(2, \mathbb{F}_q)$. With this notation, any element p' in the Levi subgroup of P can be written uniquely as

$$p' = \begin{pmatrix} A & \\ & \lambda A' \end{pmatrix}$$

with $A \in \mathrm{GL}(2, \mathbb{F}_q)$ $\lambda \in \mathbb{F}_q^{\times}$.

4.3. **Klingen.** The Klingen parabolic subgroup Q of $\mathrm{GSp}(4, \mathbb{F}_q)$ is the set of all matrices of the form

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \in \mathrm{GSp}(4, \mathbb{F}_q) \right\}.$$

Every element $g \in Q$ can be written uniquely in the form

$$
g = \begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \Delta t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},
$$

with $\Delta = ad - bc \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q^\times$, and $\kappa, \lambda, \mu \in \mathbb{F}_q$. The order of Q is therefore $q^4(q^2 - 1)(q - 1)^2$. The multiplier of g given above is $\lambda(g) = \Delta$.

4.4. **Conjugacy classes of B, P, and Q.** The following table indicates which conjugacy classes of $\mathrm{GSp}(4, \mathbb{F}_q)$ have a non-empty intersection with the Borel, Klingen parabolic, or Siegel parabolic subgroup. The numbers in the table in the columns B, P, and Q indicate if a conjugacy class intersects the respective subgroup and how many splittings it has. Blank entries indicate empty intersections.

Table 2: Conjugacy classes of B, P, and Q

Class	B	P	Q	Class	B	P	Q
$A_1(k)$	1	1	1	$C_1(i,k)$	4	3	2
$A_2(k)$	2	1	2	$C_{21}(i,k)$	8	4	4
$A_{31}(k)$	3	2	2	$C_{22}(i,k)$		2	
$A_{32}(k)$	1	1	1	$C_3(i,k)$	4	3	2
$A_5(k)$	1	1	1	$C_4(i,k)$	4	2	3
$B_{11}(k)$	2	1	2	$C_5(i,k)$	4	2	3
$B_{12}(k)$		1		$C_6(i,j,k)$	8	4	4
$B_{21}(k)$	4	3	2	$D_1(i,k)$			2
$B_{22}(k)$		1		$D_2(i,k)$		1	
$B_3(k)$	2	1	2	$D_{31}(i,k)$		2	
$B_{41}(k)$	2	1	2	$D_{32}(i,k)$			
$B_{42}(k)$	2	1	2	$D_4(i,j,k)$		2	
$B_{43}(k)$		1		$D_5(i,k)$		1	
$B_{44}(k)$		1		$D_6(i,k)$			1
$B_{51}(k)$	4	3	2	$D_7(i,j,k)$			
$B_{52}(k)$		1		$D_8(i,k)$			1
				$D_9(i,k)$			

5. Parabolic induction

The character values of parabolically induced representations supported on the subgroups B, P, and Q can now be computed. If a conjugacy class is not listed in the following character tables, the character value on that class is 0.

5.1. **Borel.** Let χ_1, χ_2, and σ be characters of the multiplicative group $\mathbb{F}_q^\times$. Define a character on the Borel subgroup B by

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \longmapsto \chi_1(a)\chi_2(b)\sigma(c).$$

The character of this representation is given by $\chi_1(a)\chi_2(b)\sigma(c)$. This representation is induced to obtain a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, denoted by $\chi_1 \times \chi_2 \rtimes \sigma$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow \mathbb{C}$$

satisfying

$$f(hg) = \chi_1(a)\chi_2(b)\sigma(c)f(g), \quad \text{for all } h = \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \in B.$$

The group action is by right translation. The central character of $\chi_1 \times \chi_2 \rtimes \sigma$ is $\chi_1\chi_2\sigma^2$. The character table of $\chi_1 \times \chi_2 \rtimes \sigma$ is the following.

Table 3: $\chi_1 \times \chi_2 \rtimes \sigma$ character values

Class	$\chi_1 \times \chi_2 \rtimes \sigma$ **character value**
$A_1(k)$	$(q^2 + 1)(q + 1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_2(k)$	$(q + 1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_{31}(k)$	$(3q + 1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_{32}(k)$	$(q + 1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$A_5(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})$
$B_{11}(k)$	$(q + 1)^2\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1) + \chi_2(-1)\Big)$
$B_{21}(k)$	$(q + 1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k})\Big(1 + \chi_1(-1)\chi_2(-1) + \chi_1(-1) + \chi_2(-1)\Big)$
$B_3(k)$	$(q + 1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1) + \chi_2(-1)\Big)$
$B_{41}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1) + \chi_2(-1)\Big)$
$B_{42}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(-1) + \chi_2(-1)\Big)$

Table 3 – Continued

Class	$\chi_1 \times \chi_2 \rtimes \sigma$ **character value**
$B_{51}(k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k})\Big(1 + \chi_1(-1)\chi_2(-1) + \chi_1(-1) + \chi_2(-1)\Big)$
$C_1(i,k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i})\Big(1 + \chi_1(\gamma^i)\chi_2(\gamma^i) + \chi_1(\gamma^i) + \chi_2(\gamma^i)\Big)$
$C_{21}(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(-\gamma^{2k+i})\Big(\chi_1(-1) + \chi_2(-1) +$ $\chi_1(-1)\chi_2(-1)(\chi_1(\gamma^i) + \chi_2(\gamma^i)) + (\chi_1(-1) +$ $\chi_2(-1))\chi_1(\gamma^i)\chi_2(\gamma^i) + \chi_1(\gamma^i) + \chi_2(\gamma^i)\Big)$
$C_3(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i})\Big(1 + \chi_1(\gamma^i)\chi_2(\gamma^i) + \chi_1(\gamma^i) + \chi_2(\gamma^i)\Big)$
$C_4(i,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(\gamma^i) + \chi_1(\gamma^{-i}) + \chi_2(\gamma^i) + \chi_2(\gamma^{-i})\Big)$
$C_5(i,k)$	$(q+1)\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k})\Big(\chi_1(\gamma^i) + \chi_1(\gamma^{-i}) + \chi_2(\gamma^i) + \chi_2(\gamma^{-i})\Big)$
$C_6(i,j,k)$	$\chi_1(\gamma^k)\chi_2(\gamma^k)\sigma(\gamma^{2k+i+j})\Big(\chi_1(\gamma^i) + \chi_1(\gamma^j) + \chi_2(\gamma^i) + \chi_2(\gamma^j) +$ $\chi_1(\gamma^{i+j})(\chi_2(\gamma^i) + \chi_2(\gamma^j)) + \chi_2(\gamma^{i+j})\big(\chi_1(\gamma^i) + \chi_1(\gamma^j)\big)\Big)$

5.2. Siegel. Let (π, V) be an irreducible representation of $\mathrm{GL}(2, \mathbb{F}_q)$, and let σ be a character of $\mathbb{F}_q^\times$. Define a representation of the Siegel parabolic subgroup P on V by

$$\begin{pmatrix} A & * \\ & cA' \end{pmatrix} \longmapsto \sigma(c)\pi(A).$$

The character of this representation is given by $\sigma(c)\chi_\pi(A)$, where χ_π is the character of π. This representation is induced to obtain a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, denoted by $\pi \rtimes \sigma$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow V$$

satisfying

$$f(hg) = \sigma(c)\pi(A)f(g), \quad \text{for all } h = \begin{pmatrix} A & * \\ & cA' \end{pmatrix} \in P.$$

The group action is by right translation. If π has central character ω_π, then the central character of $\pi \rtimes \sigma$ is $\omega_\pi\sigma^2$. The character table of $\pi \rtimes \sigma$ is the following.

Table 4: $\pi \rtimes \sigma$ character values

Class	$\pi \rtimes \sigma$ **character value**
$A_1(k)$	$(q^2+1)(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_2(k)$	$(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_{31}(k)$	$\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})+2q\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})\Big)$
$A_{32}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_5(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})$
$B_{11}(k)$	$(q+1)^2\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$
$B_{12}(k)$	$(q^2+1)\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix})$
$B_{21}(k)$	$\sigma(-\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})+\chi_\pi(\begin{pmatrix} -\gamma^k & \\ & -\gamma^k \end{pmatrix})$ $+(q+1)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})\Big)$
$B_{22}(k)$	$(q+1)\sigma(-\gamma^{2k+1})\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix})$
$B_3(k)$	$(q+1)\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$
$B_{41}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$
$B_{42}(k)$	$\sigma(\gamma^{2k})\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$

Table 4 – Continued

Class	$\pi \rtimes \sigma$ character value
$B_{43}(k)$	$\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix}))$
$B_{44}(k)$	$\sigma(\gamma^{2k+1})\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix}))$
$B_{51}(k)$	$\sigma(-\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix}))$ $+ \chi_\pi(\begin{pmatrix} -\gamma^k & 1 \\ & -\gamma^k \end{pmatrix}))\Big)$
$B_{52}(k)$	$\sigma(-\gamma^{2k+1})\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix}))$
$C_1(i,k)$	$\sigma(\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^{k+i} & \\ & \gamma^{k+i} \end{pmatrix}))$ $+(q+1)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix}))\Big)$
$C_{21}(i,k)$	$\sigma(-\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^{k+i} & \\ & -\gamma^{k+i} \end{pmatrix}))$ $+\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix})) + \chi_\pi(\begin{pmatrix} -\gamma^k & \\ & -\gamma^{k+i} \end{pmatrix}))\Big)$
$C_{22}(i,k)$	$\sigma(-\gamma^{2k+i+1})\Big(\chi_\pi(\begin{pmatrix} \gamma^{k+1/2} & \\ & -\gamma^{k+1/2} \end{pmatrix})) +$ $\chi_\pi(\begin{pmatrix} \gamma^{k+i+1/2} & \\ & -\gamma^{k+i+1/2} \end{pmatrix}))\Big)$
$C_3(i,k)$	$\sigma(\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^{k+i} & 1 \\ & \gamma^{k+i} \end{pmatrix})) +$ $\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix}))\Big)$
$C_4(i,k)$	$\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k-i} \end{pmatrix}))\Big)$
$C_5(i,k)$	$(q+1)\sigma(\gamma^{2k})\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix})) + \chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k-i} \end{pmatrix}))\Big)$

Table 4 – Continued

Class	$\pi \rtimes \sigma$ **character value**
$C_6(i,j,k)$	$\sigma(\gamma^{2k+i+j})\Big(\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+j}\end{pmatrix}))$ $+\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k+i+j}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+j} & \\ & \gamma^{k+i+j}\end{pmatrix}))$
$D_2(i,k)$	$(q+1)\sigma(\gamma^{2k+i})\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$
$D_{31}(i,k)$	$\sigma(-\gamma^{2k+i})\Big(\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix}) + \chi_\pi(\begin{pmatrix}-\gamma^k\theta^i & \\ & -\gamma^k\theta^{qi}\end{pmatrix}))$
$D_4(i,j,k)$	$\sigma(\gamma^{2k+i+j})\Big(\chi_\pi(\begin{pmatrix}\gamma^k\theta^{qi} & \\ & \gamma^k\theta^i\end{pmatrix}) + \chi_\pi(\begin{pmatrix}\gamma^{k+j}\theta^{qi} & \\ & \gamma^{k+j}\theta^i\end{pmatrix}))$
$D_5(i,k)$	$\sigma(\gamma^{2k+i})\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$

5.3. **Klingen.** Let χ be a character of $\mathbb{F}_q^\times$, and let (π, V) be an irreducible representation of $\mathrm{GL}(2, \mathbb{F}_q)$. Define a representation of the Klingen parabolic subgroup Q on V by

$$\begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \longmapsto \chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}), \quad \text{where } \Delta = ad - bc.$$

The character of this representation is given by $\chi(t)\chi_\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$, where χ_π is the character of π. This representation is induced to obtain a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, denoted by $\chi \rtimes \pi$. The standard model of this representation is the space of functions

$$f : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow V$$

satisfying

$$f(hg) = \chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})f(g), \quad \text{for all } h = \begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \in Q.$$

The group action is by right translation. If π has central character ω_π, then the central character of $\chi \rtimes \pi$ is $\chi\omega_\pi$. The character table of $\chi \rtimes \pi$ is the following.

Table 5: $\chi \rtimes \pi$ character values

Conjugacy class	$\chi \rtimes \pi$ **character value**
$A_1(k)$	$(q^2+1)(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix})$
$A_2(k)$	$\chi(\gamma^k)\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix}) + q(q+1)\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})\Big)$
$A_{31}(k)$	$2\chi(\gamma^k)\Big(\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix}) + \dfrac{(q-1)}{2}\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})\Big)$
$A_{32}(k)$	$(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})$
$A_5(k)$	$\chi(\gamma^k)\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix})$
$B_{11}(k)$	$(q+1)\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix}) + \chi_\pi(\begin{pmatrix} -\gamma^k & \\ & -\gamma^k \end{pmatrix})\Big)$
$B_{21}(k)$	$(q+1)\chi(\gamma^k)\Big(1+\chi(-1)\Big)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$
$B_3(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^k \end{pmatrix}) + (q+1)\chi_\pi(\begin{pmatrix} -\gamma^k & 1 \\ & -\gamma^k \end{pmatrix})\Big)$
$B_{41}(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix}) + \chi_\pi(\begin{pmatrix} -\gamma^k & 1 \\ & -\gamma^k \end{pmatrix})\Big)$
$B_{42}(k)$	$\chi(\gamma^k)\Big(\chi(-1)\chi_\pi(\begin{pmatrix} \gamma^k & 1 \\ & \gamma^k \end{pmatrix}) + \chi_\pi(\begin{pmatrix} -\gamma^k & 1 \\ & -\gamma^k \end{pmatrix})\Big)$
$B_{51}(k)$	$\chi(\gamma^k)\Big(1+\chi(-1)\Big)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & -\gamma^k \end{pmatrix})$
$C_1(i,k)$	$(q+1)\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix} \gamma^k & \\ & \gamma^{k+i} \end{pmatrix})$

Table 5 – Continued

Conjugacy class	$\chi \rtimes \pi$ character value
$C_{21}(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(-1))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & -\gamma^{k+i}\end{pmatrix})) \\ +(1+\chi(-\gamma^i))\chi_\pi(\begin{pmatrix}-\gamma^k & \\ & \gamma^{k+i}\end{pmatrix}))\Big)$
$C_3(i,k)$	$\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i}\end{pmatrix})$
$C_4(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(\gamma^{-i}))\chi_\pi(\begin{pmatrix}\gamma^k & 1 \\ & \gamma^k\end{pmatrix})+\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k-i}\end{pmatrix}))\Big)$
$C_5(i,k)$	$\chi(\gamma^k)\Big((\chi(\gamma^i)+\chi(\gamma^{-i}))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^k\end{pmatrix})+(q+ \\ 1)\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k-i}\end{pmatrix}))\Big)$
$C_6(i,j,k)$	$\chi(\gamma^k)\Big((1+\chi(\gamma^{i+j}))\chi_\pi(\begin{pmatrix}\gamma^{k+i} & \\ & \gamma^{k+j}\end{pmatrix})+ \\ (\chi(\gamma^i)+\chi(\gamma^j))\chi_\pi(\begin{pmatrix}\gamma^k & \\ & \gamma^{k+i+j}\end{pmatrix}))\Big)$
$D_1(i,k)$	$\chi(\gamma^k)\Big(1+\chi(\gamma^i)\Big)\chi_\pi(\begin{pmatrix}\gamma^k\theta^i & \\ & \gamma^k\theta^{qi}\end{pmatrix})$
$D_6(i,k)$	$(q+1)\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k\eta^i & \\ & \gamma^k\eta^{-i}\end{pmatrix})$
$D_8(i,k)$	$\chi(\gamma^k)\chi_\pi(\begin{pmatrix}\gamma^k\eta^i & \\ & \gamma^k\eta^{-i}\end{pmatrix})$

6. Generic representations

Recall the subgroup $N_{\mathrm{GSp}(4)}$ of $\mathrm{GSp}(4,\mathbb{F}_q)$ defined by

$$N_{\mathrm{GSp}(4)} = \left\{ \begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} \in \mathrm{GSp}(4,\mathbb{F}_q) \right\}.$$

Let ψ_1 and ψ_2 be non-trivial characters of $\mathbb{F}_q$ and let ψ_N be the character of $N_{\mathrm{GSp}(4)}$ defined by

$$\psi_N \begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} = \psi_1(x)\psi_2(y).$$

This defines a representation of $N_{\mathrm{GSp}(4)}$. Denote the representation of $\mathrm{GSp}(4, \mathbb{F}_q)$ induced from ψ_N by $\mathcal{G}$ and its character by $\chi_{\mathcal{G}}$.

If (π, V) is an irreducible representation that can be embedded into $\mathcal{G}$, we call its image a *Whittaker model of* π and say that π is *generic*. A Whittaker model of π is then a space $\mathcal{W}(\pi)$ of functions

$$W : \mathrm{GSp}(4, \mathbb{F}_q) \longrightarrow \mathbb{C}$$

having the property that

$$W\left(\begin{pmatrix} 1 & y & * & * \\ & 1 & x & * \\ & & 1 & -y \\ & & & 1 \end{pmatrix} g \right) = \psi_1(x)\psi_2(y)W(g).$$

Genericity of an irreducible representation π of $\mathrm{GSp}(4, \mathbb{F}_q)$ is easy to determine using character theory. Indeed, for a particular irreducible non-cuspidal representation π of $\mathrm{GSp}(4, \mathbb{F}_q)$ with character χ_π, one computes the inner product $(\chi_\pi, \chi_{\mathcal{G}})$. If $(\chi_\pi, \chi_{\mathcal{G}}) = 0$, then π is not generic. If $(\chi_\pi, \chi_{\mathcal{G}}) \neq 0$, then π is generic. Moreover, when $(\chi_\pi, \chi_{\mathcal{G}}) \neq 0$, then $(\chi_\pi, \chi_{\mathcal{G}}) = 1$, i.e., Whittaker models are unique. The uniqueness of Whittaker models is known in general, but it is verified computationally.

Therefore, to determine genericity, one computes the conjugacy classes of $N_{\mathrm{GSp}(4)}$ and the character table of $\mathcal{G}$. The character table of $\mathcal{G}$ is the following.

Table 6: $\mathcal{G}$ character values

Conjugacy class	$\mathcal{G}$ character value
$A_1(q-1)$	$(q^4 - 1)(q^2 - 1)(q - 1)$
$A_2(q-1)$	$-(q^2 - 1)(q - 1)$
$A_{31}(q-1)$	$-(q^2 - 1)(q - 1)$
$A_{32}(q-1)$	$-(q^2 - 1)(q - 1)$
$A_5(q-1)$	$q - 1$

We calculate $(\chi_{\mathcal{G}}, \chi_{\mathcal{G}}) = q^2(q - 1)$, which means that there are precisely $q^2(q - 1)$ irreducible generic representations of $\mathrm{GSp}(4, \mathbb{F}_q)$.

7. Irreducible characters

All of the irreducible characters of $\mathrm{Sp}(4, \mathbb{F}_q)$ were determined by Srinivasan in [18]. Her list of characters is used to determine all of the irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$. A complete list of irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ will help to determine the irreducible constituents of the parabolically induced representations defined on the Borel, the Siegel parabolic, and the Klingen parabolic subgroup. These constituents are the *non-cuspidal* characters.

The irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ are determined as follows. Each irreducible representation of $\mathrm{Sp}(4, \mathbb{F}_q)$ is extended to a representation of $\mathrm{GSp}(4, \mathbb{F}_q)$, where $\mathrm{GSp}(4, \mathbb{F}_q)^+ := Z \cdot \mathrm{Sp}(4, \mathbb{F}_q)$. Note that $\mathrm{GSp}(4, \mathbb{F}_q)^+$ is an index two subgroup of $\mathrm{GSp}(4, \mathbb{F}_q)$ and that $Z \cap \mathrm{Sp}(4, \mathbb{F}_q) = \pm I$, where I is the identity of $\mathrm{GSp}(4, \mathbb{F}_q)$. For an irreducible representation π of $\mathrm{Sp}(4, \mathbb{F}_q)$, an irreducible representation π^+ of $\mathrm{GSp}(4, \mathbb{F}_q)^+$ is constructed by defining $\pi^+(z \cdot g) := \alpha(z)\pi(g)$, where α is a character of Z, hence a character of $\mathbb{F}_q^\times$. By Schur's Lemma, elements of Z act as scalars on vectors in the space of π. To ensure that this new representation is well-defined, it is required that $\alpha(\pm 1)$ acts as $\pi(\pm I)$ on the space of π. The character of this new representation is $\alpha(z)\chi_\pi(g)$, where χ_π is the character of π. This representation is then induced to $\mathrm{GSp}(4, \mathbb{F}_q)$. The induced representation is either irreducible or has precisely two irreducible constituents.

The group $\mathrm{GSp}(4, \mathbb{F}_q)^+$ only has elements with square multipliers and so the induced character takes the value 0 on the non-square multiplier classes of $\mathrm{GSp}(4, \mathbb{F}_q)$. If the induced character decomposes into two constituents, then the sum of the values of the constituent characters on the non-square multiplier classes is 0. The values of the constituent characters on the square multiplier classes are half the values of the induced character on those classes.

The list of all of the nontrivial irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$ is now given using the notation of Srinivasan in [18] obtained using the method described above. Let $\alpha : \mathbb{F}_q^\times \longrightarrow \mathbb{C}^\times$ be a character. The notation used is described as follows. Let χ be an irreducible character of $\mathrm{Sp}(4, \mathbb{F}_q)$. $\alpha\chi$ will denote the irreducible character of $\mathrm{GSp}(4, \mathbb{F}_q)^+$ obtained by extending χ to $\mathrm{GSp}(4, \mathbb{F}_q)^+$ by a character α on its center. The character of $\mathrm{GSp}(4, \mathbb{F}_q)$ induced from $\alpha\chi$ is denoted by $\mathrm{Ind}(\alpha\chi)$ or simply by $\mathrm{Ind}(\chi)$ if α is the trivial character.

The table below lists each of these characters, the value of $\alpha(-1)$, notation for the irreducible constituents of the induced character, and the dimension of each constituent. In the cases where the induced character decomposes, say into χ_a and χ_b, we have that $\chi_b = \xi\chi_a$, where $\xi : \mathbb{F}_q^\times \longrightarrow \mathbb{C}$ is the character defined by $\xi(\gamma) = -1$. Genericity of a character is indicated by a $\bullet$ in the "g" column. The abbreviation $t = \frac{1}{2}(q - 1)$ is also used.

Table 7: Irreducible characters of $\mathrm{GSp}(4, \mathbb{F}_q)$

Character	$\alpha(-1)$	Constituents	Dimension	g
$\mathrm{Ind}(\alpha\chi_1(n))$ $n \in R_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\chi_1(n))_a$	$(q^2-1)^2$	•
		$\mathrm{Ind}(\alpha\chi_1(n))_b$	$(q^2-1)^2$	•
$\mathrm{Ind}(\alpha\chi_2(n))$ $n \in R_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\chi_2(n))_a$	q^4-1	•
		$\mathrm{Ind}(\alpha\chi_2(n))_b$	q^4-1	•
$\mathrm{Ind}(\alpha\chi_3(n,m))$ $n, m \in T_1, n < m$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_3(n,m))_a$	$(q^2+1)(q+1)^2$	•
		$\mathrm{Ind}(\alpha\chi_3(n,m))_b$	$(q^2+1)(q+1)^2$	•
$\mathrm{Ind}(\alpha\chi_4(n,m))$ $n, m \in T_2, n \neq m$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_4(n,m))_a$	$(q^2+1)(q-1)^2$	•
		$\mathrm{Ind}(\alpha\chi_4(n,m))_b$	$(q^2+1)(q-1)^2$	•
$\mathrm{Ind}(\alpha\chi_5(n,m))$ $n \in T_2, m \in T_1$	$(-1)^{n+m}$	$\mathrm{Ind}(\alpha\chi_5(n,m))_a$	q^4-1	•
		$\mathrm{Ind}(\alpha\chi_5(n,m))_b$	q^4-1	•
$\mathrm{Ind}(\alpha\chi_6(n))$ $n \in T_2$	1	$\mathrm{Ind}(\alpha\chi_6(n))_a$	$(q^2+1)(q-1)$	
		$\mathrm{Ind}(\alpha\chi_6(n))_b$	$(q^2+1)(q-1)$	
$\mathrm{Ind}(\alpha\chi_7(n))$ $n \in T_2$	1	$\mathrm{Ind}(\alpha\chi_7(n))_a$	$q(q^2+1)(q-1)$	•
		$\mathrm{Ind}(\alpha\chi_7(n))_b$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\chi_8(n))$ $n \in T_1$	1	$\mathrm{Ind}(\alpha\chi_8(n))_a$	$(q^2+1)(q+1)$	
		$\mathrm{Ind}(\alpha\chi_8(n))_b$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\chi_9(n))$ $n \in T_1$	1	$\mathrm{Ind}(\alpha\chi_9(n))_a$	$q(q^2+1)(q+1)$	•
		$\mathrm{Ind}(\alpha\chi_9(n))_b$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\xi_1(n))$ $n \in T_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_1(n))_a$	$(q^2+1)(q-1)$	
		$\mathrm{Ind}(\alpha\xi_1(n))_b$	$(q^2+1)(q-1)$	
$\mathrm{Ind}(\alpha\xi_1'(n))$ $n \in T_2$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_1'(n))_a$	$q(q^2+1)(q-1)$	•
		$\mathrm{Ind}(\alpha\xi_1'(n))_b$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\xi_{21}(n))$ $n \in T_2$	$(-1)^{n+t}$	$\mathrm{Ind}(\alpha\xi_{21}(n))$	q^4-1	•
$\mathrm{Ind}(\alpha\xi_{21}'(n))$ $n \in T_2$	$(-1)^{n+t+1}$	$\mathrm{Ind}(\alpha\xi_{21}'(n))$	$(q^2+1)(q-1)^2$	•

Table 7 – Continued

Character	$\alpha(-1)$	Constituents	Dimension	g
$\mathrm{Ind}(\alpha\xi_3(n))$ $n \in T_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_3(n))_a$	$(q^2+1)(q+1)$	
		$\mathrm{Ind}(\alpha\xi_3(n))_b$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\xi_3'(n))$ $n \in T_1$	$(-1)^n$	$\mathrm{Ind}(\alpha\xi_3'(n))_a$	$q(q^2+1)(q+1)$	•
		$\mathrm{Ind}(\alpha\xi_3'(n))_b$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\xi_{41}(n))$ $n \in T_1$	$(-1)^{n+t}$	$\mathrm{Ind}(\alpha\xi_{41}(n))$	$(q^2+1)(q+1)^2$	•
$\mathrm{Ind}(\alpha\xi_{41}'(n))$ $n \in T_1$	$(-1)^{n+t+1}$	$\mathrm{Ind}(\alpha\xi_{41}'(n))$	q^4-1	•
$\mathrm{Ind}(\alpha\Phi_1)$	$(-1)^{t+1}$	$\mathrm{Ind}(\alpha\Phi_1)$	$(q^2+1)(q-1)$	
$\mathrm{Ind}(\alpha\Phi_3)$	$(-1)^{t+1}$	$\mathrm{Ind}(\alpha\Phi_3)$	$q(q^2+1)(q-1)$	•
$\mathrm{Ind}(\alpha\Phi_5)$	$(-1)^t$	$\mathrm{Ind}(\alpha\Phi_5)$	$(q^2+1)(q+1)$	
$\mathrm{Ind}(\alpha\Phi_7)$	$(-1)^t$	$\mathrm{Ind}(\alpha\Phi_7)$	$q(q^2+1)(q+1)$	•
$\mathrm{Ind}(\alpha\Phi_9)$	1	$\mathrm{Ind}(\alpha\Phi_9)_a$	$q(q^2+1)$	
		$\mathrm{Ind}(\alpha\Phi_9)_b$	$q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_1)$	1	$\mathrm{Ind}(\alpha\theta_1)$	$q^2(q^2+1)$	•
$\mathrm{Ind}(\alpha\theta_3)$	1	$\mathrm{Ind}(\alpha\theta_3)$	q^2+1	
$\mathrm{Ind}(\alpha\theta_5)$	-1	$\mathrm{Ind}(\alpha\theta_5)$	$q^2(q^2-1)$	•
$\mathrm{Ind}(\alpha\theta_7)$	-1	$\mathrm{Ind}(\alpha\theta_7)$	q^2-1	
$\mathrm{Ind}(\alpha\theta_9)$	1	$\mathrm{Ind}(\alpha\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
		$\mathrm{Ind}(\alpha\theta_9)_b$	$\frac{1}{2}q(q+1)^2$	
$\mathrm{Ind}(\alpha\theta_{10})$	1	$\mathrm{Ind}(\alpha\theta_{10})_a$	$\frac{1}{2}q(q-1)^2$	
		$\mathrm{Ind}(\alpha\theta_{10})_b$	$\frac{1}{2}q(q-1)^2$	
$\mathrm{Ind}(\alpha\theta_{11})$	1	$\mathrm{Ind}(\alpha\theta_{11})_a$	$\frac{1}{2}q(q^2+1)$	
		$\mathrm{Ind}(\alpha\theta_{11})_b$	$\frac{1}{2}q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_{12})$	1	$\mathrm{Ind}(\alpha\theta_{12})_a$	$\frac{1}{2}q(q^2+1)$	
		$\mathrm{Ind}(\alpha\theta_{12})_b$	$\frac{1}{2}q(q^2+1)$	
$\mathrm{Ind}(\alpha\theta_{13})$	1	$\mathrm{Ind}(\alpha\theta_{13})_a$	q^4	•
		$\mathrm{Ind}(\alpha\theta_{13})_b$	q^4	•

There are $(q^2 + 2q + 4)(q-1)$ irreducible characters and $q^2(q-1)$ irreducible generic characters.

7.1. Irreducible non-cuspidal representations.

The following table contains information on the irreducible non-cuspidal representations of the group $\mathrm{GSp}(4, \mathbb{F}_q)$. Representations in the same group, denoted by a roman numeral, are constituents of the same induced representation. The reader should notice that the group notation IV is not present. The representations associated to those given in the groups IV and VI in [14] are combined in the group IV*. The "g" column in the table indicates whether a representation is generic.

The irreducible constituents in the table are sometimes written as twists of an irreducible character rather than in the form that is in the table of irreducible characters above. For example, the Va constituent is written as $\sigma\mathrm{Ind}(\theta_1)$, where θ_1 is an irreducible character of $\mathrm{Sp}(4, \mathbb{F}_q)$ which is extended to $\mathrm{GSp}(4, \mathbb{F}_q)^+$ using the trivial character on the center, induced to $\mathrm{GSp}(4, \mathbb{F}_q)$, then twisted by the character σ. This constituent can also be written as $\mathrm{Ind}(\sigma^2\theta_1)$. The other such constituents can be written using the notation in the irreducible character table similarly.

Group I. These are the irreducible representations obtained by parabolic induction from the Borel subgroup. More precisely, they are irreducible representations of the form $\chi_1 \times \chi_2 \rtimes \sigma$, with $\chi_1, \chi_2,$ and σ characters of $\mathbb{F}_q^\times$. Representations of this form are irreducible if and only if $\chi_1 \neq 1_{\mathbb{F}_q^\times}, \chi_2 \neq 1_{\mathbb{F}_q^\times}$ and $\chi_1 \neq \chi_2^{\pm 1}$, where $1_{\mathbb{F}_q^\times}$ is the trivial character of $\mathbb{F}_q^\times$.

Group II. Let χ be a character of $\mathbb{F}_q^\times$ with $\chi^2 \neq 1_{\mathbb{F}_q^\times}$. Then the induced representation $\chi \times \chi \rtimes \sigma$ decomposes into two irreducible constituents

$$\mathrm{IIa} : \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \quad \text{and} \quad \mathrm{IIb}: \chi 1_{\mathrm{GL}(2)} \rtimes \sigma,$$

where $\mathrm{St}_{\mathrm{GL}(2)}$ denotes the Steinberg representation of $\mathrm{GL}(2, \mathbb{F}_q)$ and $1_{\mathrm{GL}(2)}$ denotes the trivial representation of $\mathrm{GL}(2, \mathbb{F}_q)$. $\mathrm{St}_{\mathrm{GL}(2)}$ and $1_{\mathrm{GL}(2)}$ are obtained as the irreducible constituents of the induced representation $1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times}$ of $\mathrm{GL}(2, \mathbb{F}_q)$.

Group III. Let χ be a character of $\mathbb{F}_q^\times$ such that $\chi \neq 1_{\mathbb{F}_q^\times}$. Then $\chi \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$ decomposes into two irreducible constituents

$$\mathrm{IIIa} : \chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} \quad \text{and} \quad \mathrm{IIIb}: \chi \rtimes \sigma 1_{\mathrm{GSp}(2)},$$

where $\mathrm{St}_{\mathrm{GSp}(2)}$ denotes the Steinberg representation of $\mathrm{GSp}(2, \mathbb{F}_q)$ and $1_{\mathrm{GSp}(2)}$ denotes the trivial representation of $\mathrm{GSp}(2, \mathbb{F}_q)$. $\mathrm{St}_{\mathrm{GSp}(2)}$ and $1_{\mathrm{GSp}(2)}$ are obtained as the constituents of the induced representation $1_{\mathbb{F}_q^\times} \rtimes 1_{\mathbb{F}_q^\times}$ of $\mathrm{GSp}(2, \mathbb{F}_q)$.

Group V. Let ξ be a non-trivial quadratic character of $\mathbb{F}_q^\times$. Then $\xi \times \xi \rtimes \sigma$ decomposes into four irreducible constituents

$$\mathrm{Va}: \sigma\mathrm{Ind}(\theta_1) \quad \mathrm{Vb}: \sigma\mathrm{Ind}(\Phi_9)_a$$
$$\mathrm{Vc}: \sigma\mathrm{Ind}(\Phi_9)_b \quad \mathrm{Vd}: \sigma\mathrm{Ind}(\theta_3).$$

Group VI.* $1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$ decomposes into six irreducible constituents

$$\mathrm{VI^*a}: \sigma\mathrm{St}_{\mathrm{GSp}(4)} \quad \mathrm{VI^*b}: \sigma\mathrm{Ind}(\theta_9)_a$$
$$\mathrm{VI^*c}: \sigma\mathrm{Ind}(\theta_9)_a \quad \mathrm{VI^*d}: \sigma\mathrm{Ind}(\theta_{11})_a$$
$$\mathrm{VI^*e}: \sigma\mathrm{Ind}(\theta_{12})_a \quad \mathrm{VI^*f}: \sigma 1_{\mathrm{GSp}(4)},$$

where $\mathrm{St}_{\mathrm{GSp}(4)} = \mathrm{Ind}(\theta_{13})$ is the Steinberg representation of $\mathrm{GSp}(4, \mathbb{F}_q)$ and $1_{\mathrm{GSp}(4)}$ is the trivial representation of $\mathrm{GSp}(4, \mathbb{F}_q)$.

Group VII. These are the irreducible representations of the form $\chi \rtimes \pi$, where π is an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ and χ is a character of $\mathbb{F}_q^\times$.

Representations of this form are irreducible if and only if $\chi \neq 1_{\mathbb{F}_q^\times}$ and $\chi \neq \xi$, where ξ is a character of order 2 such that $\xi\pi \cong \pi$.

Group VIII. Let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. Then $1_{\mathbb{F}_q^\times} \rtimes \pi$ decomposes into two irreducible constituents

$$\text{VIIIa: } \mathrm{Ind}(\omega_\pi \Phi_3) \quad \text{and} \quad \text{VIIIb: } \mathrm{Ind}(\omega_\pi \Phi_1).$$

Group IX. Let ξ be a non-trivial quadratic character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π such that $\xi\pi \cong \pi$. Then $\xi \rtimes \pi$ decomposes into two irreducible constituents

$$\text{IXa: } \mathrm{Ind}(\xi\omega_\pi \theta_5) \quad \text{and} \quad \text{IXb: } \mathrm{Ind}(\xi\omega_\pi \theta_7).$$

Group X. These are the irreducible representations of the form $\pi \rtimes \sigma$, where π is an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ and σ is a character of $\mathbb{F}_q^\times$. Representations of this form are irreducible if and only if π does not have trivial central character ω_π.

Group XI. Let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with trivial central character ω_π and let σ be a character of $\mathbb{F}_q^\times$. Then $\pi \rtimes \sigma$ decomposes into two irreducible constituents

$$\text{XIa: } \sigma\mathrm{Ind}(\chi_7(n))_a \quad \text{and} \quad \text{XIb: } \sigma\mathrm{Ind}(\chi_6(n))_a.$$

Table 8: Irreducible non-cuspidal representations

		Constituent of	**Representation**	**Dimension**	**g**
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		$(q^2+1)(q+1)^2$	•
II	a	$\chi \times \chi \rtimes \sigma$	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$q(q^2+1)(q+1)$	•
	b	$(\chi^2 \neq 1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$(q^2+1)(q+1)$	
III	a	$\chi \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$q(q^2+1)(q+1)$	•
	b	$(\chi \neq 1)$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$(q^2+1)(q+1)$	
V	a		$\sigma\mathrm{Ind}(\theta_1)$	$q^2(q^2+1)$	•
	b	$\xi \times \xi \rtimes \sigma$	$\sigma\mathrm{Ind}(\Phi_9)_a$	$q(q^2+1)$	
	c	$(\xi^2 = 1, \xi \neq 1)$	$\sigma\mathrm{Ind}(\Phi_9)_b$	$q(q^2+1)$	
	d		$\sigma\mathrm{Ind}(\theta_3)$	q^2+1	
VI*	a		$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	q^4	•
	b		$\sigma\mathrm{Ind}(\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
	c	$1_{\mathbb{F}_q^\times} \times 1_{\mathbb{F}_q^\times} \rtimes \sigma$	$\sigma\mathrm{Ind}(\theta_9)_a$	$\frac{1}{2}q(q+1)^2$	
	d		$\sigma\mathrm{Ind}(\theta_{11})_a$	$\frac{1}{2}q(q^2+1)$	
	e		$\sigma\mathrm{Ind}(\theta_{12})_a$	$\frac{1}{2}q(q^2+1)$	
	f		$\sigma 1_{\mathrm{GSp}(4)}$	1	
VII		$\chi \rtimes \pi$ (irreducible)		q^4-1	•
VIII	a	$1_{\mathbb{F}_q^\times} \rtimes \pi$	$\mathrm{Ind}(\omega_\pi \Phi_3)$	$q(q^2+1)(q-1)$	•
	b		$\mathrm{Ind}(\omega_\pi \Phi_1)$	$(q^2+1)(q-1)$	
IX	a	$\xi \rtimes \pi$	$\mathrm{Ind}(\xi\omega_\pi\theta_5)$	$q^2(q^2-1)$	•
	b	$(\xi \neq 1, \xi\pi = \pi)$	$\mathrm{Ind}(\xi\omega_\pi\theta_7)$	q^2-1	
X		$\pi \rtimes \sigma$ (irreducible)		q^4-1	•
XI	a	$\pi \rtimes \sigma$	$\sigma\mathrm{Ind}(\chi_7(n))_a$	$q(q^2+1)(q-1)$	•
	b	$(\omega_\pi = 1)$	$\sigma\mathrm{Ind}(\chi_6(n))_a$	$(q^2+1)(q-1)$	

The induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq 1$, $\chi_2 \neq 1$, and $\chi_1 \neq \chi_2^{\pm 1}$.

Several formulas are used in our proof of this table. In particular, they are used to establish reducibility criteria and to verify that a particular representation is a constituent of a non-cuspidal group of representations. Define

$$z_1 = \begin{cases} 0 & \text{if } \chi_1 \neq 1 \\ q-1 & \text{if } \chi_1 = 1 \end{cases} \qquad z_2 = \begin{cases} 0 & \text{if } \chi_2 \neq 1 \\ q-1 & \text{if } \chi_2 = 1 \end{cases}$$

$$z_3 = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_1^{-1} \\ q-1 & \text{if } \chi_1 = \chi_1^{-1} \end{cases} \qquad z_4 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_2^{-1} \\ q-1 & \text{if } \chi_2 = \chi_2^{-1} \end{cases}$$

$$z_5 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_1 \\ q-1 & \text{if } \chi_2 = \chi_1 \end{cases} \qquad z_6 = \begin{cases} 0 & \text{if } \chi_2 \neq \chi_1^{-1} \\ q-1 & \text{if } \chi_2 \neq \chi_1^{-1} \end{cases}$$

$$z_7 = \begin{cases} 0 & \text{if } \phi|_{\langle \eta \rangle} \neq 1 \\ q+1 & \text{if } \phi|_{\langle \eta \rangle} = 1 \end{cases}$$

Lemma 7.1. *Let* $\chi_1, \chi_2 : \mathbb{F}_q^\times \longrightarrow \mathbb{C}$ *be characters. Then*

$$\sum_{i \in T_1} \chi_1(\gamma^i) + \chi_1(\gamma^{-i}) = z_1 - 1 - \chi_1(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)\chi_2(\gamma^i) + \chi_1(\gamma^{-i})\chi_2(\gamma^{-i}) = z_6 - 1 - \chi_1(-1)\chi_2(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)\chi_2(\gamma^{-i}) + \chi_1(\gamma^{-i})\chi_2(\gamma^i) = z_5 - 1 - \chi_1(-1)\chi_2(-1)$$

$$\sum_{i \in T_1} \chi_1(\gamma^i)^2 + \chi_1(\gamma^{-i})^2 = z_3 - 2$$

$$\sum_{i,j \in T_1, i<j} \chi_1(\gamma^{i+j}) + \chi_1(\gamma^{i-j}) + \chi_1(\gamma^{-i+j}) + \chi_1(\gamma^{-i-j})$$

$$= \frac{1}{2}\left((z_1 - \chi_1(-1) - 1)^2 - z_3 - q + 5\right)$$

$$\sum_{i,j \in T_1, i<j} (\chi_1(\gamma^i) + \chi_1(\gamma^{-i}))(\chi_2(\gamma^i) + \chi_2(\gamma^{-i}))$$

$$= \frac{q-5}{4}(z_6 + z_5 - 2\chi_1(-1)\chi_2(-1) - 2)$$

$$\sum_{i,j \in T_1, i<j} \chi_1(\gamma^i) + \chi_1(\gamma^{-i}) + \chi_1(\gamma^j) + \chi_1(\gamma^{-j}) = \frac{q-5}{2}(z_1 - 1 - \chi_1(-1))$$

$$\sum_{i,j \in T_1, i<j} \chi_1(\gamma^i)^2 + \chi_1(\gamma^{-i})^2 + \chi_1(\gamma^j)^2 + \chi_1(\gamma^{-j})^2 = \frac{q-5}{2}(z_3 - 2)$$

Proof: Straightforward. □

Now we prove the assertions in the above table of irreducible non-cuspidal representations.

Proof: The irreducible non-cuspidal representations are supported in the Borel, the Siegel parabolic, or the Klingen parabolic. We first consider those supported in the Borel.

<u>Borel:</u> Let χ_1, χ_2, and σ be characters of $\mathbb{F}_q^\times$. As in 5.1, these characters are used to define a representation of the Borel subgroup and induced to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\chi_1 \times \chi_2 \rtimes \sigma$. From its character table, we have

$$\chi_1 \times \chi_2 \rtimes \sigma \cong \chi_2 \times \chi_1 \rtimes \sigma.$$

We also have

$$(\chi_1 \times \chi_2 \rtimes \sigma, \chi_\mathcal{G}) = 1$$

for all such representations $\chi_1 \times \chi_2 \rtimes \sigma$, indicating that exactly one irreducible constituent of $\chi_1 \times \chi_2 \rtimes \sigma$ is generic. Also,

(1)
$$(\chi_1 \times \chi_2 \rtimes \sigma, \chi_1 \times \chi_2 \rtimes \sigma) = \frac{q^2 + (z_5 + z_6 - 2)q + 1 + z_1^2 + 2z_1 z_2 + z^2 - z_5 - z_6 + z_5 z_6}{(q-1)^2}$$

which has precisely four possible values: 1,2,4,8. Equation 1 is equal to 1, or, equivalently, $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible, if and only if $\chi_1 \neq 1$, $\chi_2 \neq 1$, and $\chi_2 \neq \chi_1^{\pm 1}$.

Equation 1 is equal to 2 if one of the following holds:

1. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = \chi_1^{-1}$
2. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = \chi_1$
3. $\chi_1 \neq \chi_1^{-1}$, $\chi_2 = 1$
4. $\chi_1 = \chi_1^{-1}$, $\chi_1 \neq 1$, $\chi_2 = 1$
5. $\chi_1 = 1$, $\chi_2 \neq \chi_2^{-1}$
6. $\chi_1 = 1$, $\chi_2 = \chi_2^{-1}$, $\chi_2 \neq 1$

Equation 1 is equal to 4 if and only if $\chi_1 = \chi_2$, $\chi_2 \neq 1$, $\chi_1 = \chi_1^{-1}$.

Equation 1 is equal to 8 if and only if $\chi_1 = 1$ and $\chi_2 = 1$.

Using the character inner product, the constituents in groups II–IV* can be verified.

<u>Siegel:</u> Let σ be a character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. As in 5.2, define a representation of the Siegel parabolic subgroup and induce to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\pi \rtimes \sigma$. We have

$$(\pi \rtimes \sigma, \chi_\mathcal{G}) = 1$$

for all such representations $\pi \rtimes \sigma$, indicating that exactly one irreducible constituent of $\pi \rtimes \sigma$ is generic.

The sum of the dimensions of the irreducible constituents of $\pi \rtimes \sigma$ is $q^4 - 1$. If $\pi \rtimes \sigma$ is reducible, then by dimension considerations, the possible irreducible generic constituents are $\mathrm{Ind}(\alpha\chi_1(n))_a$, $\mathrm{Ind}(\alpha\chi_1(n))_b$, $\mathrm{Ind}(\alpha\chi_4(n,m))_a$, $\mathrm{Ind}(\alpha\chi_4(n,m))_b$, $\mathrm{Ind}(\alpha\chi_7(n))_a$, $\mathrm{Ind}(\alpha\chi_7(n))_b$, $\mathrm{Ind}(\alpha\xi_1'(n))_a$, $\mathrm{Ind}(\alpha\xi_1'(n))_b$, $\mathrm{Ind}(\alpha\xi_{21}'(n))$, $\mathrm{Ind}(\alpha\Phi_3)$, and $\mathrm{Ind}(\alpha\theta_5)$.

By adding character values, if $\pi \rtimes \sigma$ is reducible then it has precisely two irreducible constituents, ω_π is trivial, and

$$\pi \rtimes \sigma = \sigma\mathrm{Ind}(\chi_7(n))_a + \sigma\mathrm{Ind}(\chi_6(n))_a$$

for some $n \in T_2$.

<u>Klingen:</u> Let χ be a character of $\mathbb{F}_q^\times$ and let π be an irreducible cuspidal representation of $\mathrm{GL}(2, \mathbb{F}_q)$ with central character ω_π. As in 5.3, define a representation of the Klingen parabolic subgroup and induce to $\mathrm{GSp}(4, \mathbb{F}_q)$ to obtain the representation $\chi \rtimes \pi$. We have

$$(\chi \rtimes \pi, \chi_\mathcal{G}) = 1$$

for all such representations $\chi \rtimes \pi$, indicating that exactly one irreducible constituent of $\chi \rtimes \pi$ is generic.

The sum of the dimensions of the irreducible constituents of $\chi \rtimes \pi$ is $q^4 - 1$. If $\chi \rtimes \pi$ is reducible, the possible irreducible generic constituents are $\mathrm{Ind}(\alpha\chi_1(n))_a$, $\mathrm{Ind}(\alpha\chi_1(n))_b$, $\mathrm{Ind}(\alpha\chi_4(n,m))_a$, $\mathrm{Ind}(\alpha\chi_4(n,m))_b$, $\mathrm{Ind}(\alpha\chi_7(n))_a$, $\mathrm{Ind}(\alpha\chi_7(n))_b$, $\mathrm{Ind}(\alpha\xi_1'(n))_a$, $\mathrm{Ind}(\alpha\xi_1'(n))_b$, $\mathrm{Ind}(\alpha\xi_{21}'(n))$, $\mathrm{Ind}(\alpha\Phi_3)$, and $\mathrm{Ind}(\alpha\theta_5)$.

By adding character values, if $\chi \rtimes \pi$ is reducible then it has precisely two irreducible constituents and either

(1) χ is trivial and

$$1_{\mathbb{F}_q^\times} \rtimes \pi = \mathrm{Ind}(\omega_\pi\Phi_3) + \mathrm{Ind}(\omega_\pi\Phi_1)$$

or (2) $\chi = \xi$ with $\xi \neq 1$ such that $\xi\pi = \pi$ and

$$\xi \rtimes \pi = \mathrm{Ind}(\xi\omega_\pi\theta_5) + \mathrm{Ind}(\xi\omega_\pi\theta_7).$$

$\square$

7.1.1. *Decompositions for types V and VI**.

Here more information is given on the decompositions of the non-cuspidal representations supported in the Borel subgroup for types V and VI*. These decompositions can be verified with the character tables provided above using either the inner product or by adding up character values on each conjugacy class.

Group V: Constituents of $\xi \times \xi \rtimes \sigma$, where ξ is a non-trivial quadratic character.

$$\xi \times \xi \rtimes \sigma = \xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma + \xi\,1_{\mathrm{GL}(2)} \rtimes \sigma$$
$$= \xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\sigma + \xi\,1_{\mathrm{GL}(2)} \rtimes \xi\sigma.$$

Each of the four representations on the right side is reducible and has two constituents as shown in the following table.

Table 9: Group V constituents

	$\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \xi\sigma$	$\xi\,1_{\mathrm{GL}(2)} \rtimes \xi\sigma$
$\xi\,\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\mathrm{Ind}(\theta_1)$	$\sigma\mathrm{Ind}(\Phi_9)_a$
$\xi\,1_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\mathrm{Ind}(\Phi_9)_b$	$\sigma\mathrm{Ind}(\theta_3)$

Group VI*: Constituents of $1_{\mathbb{F}^\times} \times 1_{\mathbb{F}^\times} \rtimes \sigma$.

$$1_{\mathbb{F}^\times} \times 1_{\mathbb{F}^\times} \rtimes \sigma = \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma + 1_{\mathrm{GL}(2)} \rtimes \sigma$$
$$= 1_{\mathbb{F}_q^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)} + 1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}.$$

Each of the four representations on the right is reducible and has three irreducible constituents as shown in the following table. The common factor $\sigma\mathrm{Ind}(\theta_9)_a$ occurs as a constituent of each of the four representations $\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$, $1_{\mathrm{GL}(2)} \rtimes \sigma$, $1_{\mathbb{F}_q^\times} \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$, and $1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}$.

Table 10: Group VI* constituents

	$\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	(common factor)	$1_{\mathrm{GL}(2)} \rtimes \sigma$
$1_{\mathbb{F}_q^\times} \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma \mathrm{Ind}(\theta_9)_a$	$\sigma \mathrm{Ind}(\theta_{11})_a$
(common factor)	$\sigma \mathrm{Ind}(\theta_9)_a$	$-$	$\sigma \mathrm{Ind}(\theta_9)_a$
$1_{\mathbb{F}_q^\times} \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$\sigma \mathrm{Ind}(\theta_{12})_a$	$\sigma \mathrm{Ind}(\theta_9)_a$	$\sigma 1_{\mathrm{GSp}(4)}$

8. Dimension formulas

Consider the group $\mathrm{GSp}(4, F)$, where F is a non-archimedean local field of characteristic zero. Denote its ring of integers by $\mathfrak{o}$ and let $\mathfrak{p}$ be its maximal ideal. Consider only those fields F such that $\mathfrak{o}/\mathfrak{p}$ is isomorphic to the finite field $\mathbb{F}_q$ so that $\mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \cong \mathrm{GSp}(4, \mathbb{F}_q)$. Fix a generator ϖ of $\mathfrak{p}$. If x is in $F^\times$, then define $\nu(x)$ to be the unique integer such that $x = u\varpi^{\nu(x)}$ for some unit u in $\mathfrak{o}^\times$. Write $\nu(x)$ or $|x|$ for the normalized absolute value of x; thus $\nu(\varpi) = q^{-1}$.

Recall that a representation (π, V) of a group G is called *smooth* if every vector in V is fixed by an open-compact subgroup K of G and (π, V) is called *admissible* if the spaces V^K of fixed vectors under the action of an open compact subgroup K are finite dimensional for any open compact subgroup K.

The *congruence subgroup of level* $\mathfrak{p}^n$ of $\mathrm{GSp}(4, F)$, denoted by $\Gamma(\mathfrak{p}^n)$, is defined by

$$\Gamma(\mathfrak{p}^n) = \{g \in \mathrm{GSp}(4, F) : g \equiv I \,(\mathrm{mod}\,\mathfrak{p}^n)\}$$

where I is the 4×4 identity matrix.

We have the following short exact sequence

$$1 \longrightarrow \Gamma(\mathfrak{p}) \longrightarrow \mathrm{GSp}(4, \mathfrak{o}) \longrightarrow \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \longrightarrow 1$$

for the maximal compact subgroup $K = \mathrm{GSp}(4, \mathfrak{o})$ of $\mathrm{GSp}(4, F)$.

Let (π, V) be an admissible representation of $\mathrm{GSp}(4, F)$. K acts on the space $V^{\Gamma(\mathfrak{p})}$. By definition $\Gamma(\mathfrak{p})$ acts trivially in this space so there is an action of $\mathrm{GSp}(4, \mathbb{F}_q) \cong \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \cong K/\Gamma(\mathfrak{p})$.

For an admissible non-supercuspidal representation of $\mathrm{GSp}(4, F)$, the dimensions of the spaces $V^{\Gamma(\mathfrak{p})}$ can be determined by looking at their finite group analogues. For example, consider the irreducible non-supercuspidal representation $\chi_1 \times \chi_2 \rtimes \sigma$ given in [15]. The standard model for this representation is the space of functions

$$f : \mathrm{GSp}(4, F) \longrightarrow \mathbb{C}$$

satisfying

$$f(\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} g) = |a^2 b||c|^{-3/2} \chi_1(a) \chi_2(b) \sigma(c) f(g),$$

with group action by right translation. χ_1, χ_2, and σ are characters of the multiplicative group of the field F, i.e., $\chi_1, \chi_2, \sigma : F^\times \longrightarrow \mathbb{C}^\times$. Let us assume that these characters are trivial on $1 + \mathfrak{p}$. We restrict these characters to $\mathfrak{o}^\times$ to get characters on the multiplicative group of the finite field $\mathbb{F}_q$, i.e.,

$$\tilde{\chi}_i := \chi_i|_{\mathfrak{o}^\times}, \tilde{\sigma} := \sigma|_{\mathfrak{o}^\times} : \mathbb{F}_q^\times \cong (\mathfrak{o}/\mathfrak{p})^\times = \mathfrak{o}^\times/(1 + \mathfrak{p}) \longrightarrow \mathbb{C}^\times.$$

The subspace of this representation $V^{\Gamma(\mathfrak{p})}$ of vectors fixed under the action of the congruence subgroup $\Gamma(\mathfrak{p})$

$$V^{\Gamma(\mathfrak{p})} = \{f : f(gk) = f(g) \text{ for all } k \in \Gamma(\mathfrak{p})\}$$

is isomorphic to the space $\tilde{V}^{\Gamma(\mathfrak{p})}$ of functions $f : K \longrightarrow \mathbb{C}$ satisfying the above property. These functions are then functions $f : K/\Gamma(\mathfrak{p}) \longrightarrow \mathbb{C}$, hence functions $f : \mathrm{GSp}(4, \mathfrak{o}/\mathfrak{p}) \longrightarrow \mathbb{C}$. So f can be considered to be a function on the finite group $\mathrm{GSp}(4, \mathbb{F}_q)$ satisfying

$$f(\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} g) = \tilde{\chi}_1(a)\tilde{\chi}_2(b)\tilde{\sigma}(c)f(g).$$

So

$$(\chi_1 \times \chi_2 \rtimes \sigma)^{\Gamma(\mathfrak{p})} \cong \tilde{\chi}_1 \times \tilde{\chi}_2 \rtimes \tilde{\sigma}.$$

Thus the dimension of $(\chi_1 \times \chi_2 \rtimes \sigma)^{\Gamma(\mathfrak{p})}$ is $(q^2 + 1)(q + 1)^2$.

The non-supercuspidal representations obtained from parabolic induction on the Siegel and Klingen parabolic groups also decend to non-cuspidal representations in the finite group case under appropriate assumptions on the characters σ and χ and the GL(2) representation π, i.e., σ and χ are trivial on $1 + \mathfrak{p}$, and π has non-zero $\Gamma(\mathfrak{p})$–fixed vectors for the principal congruence subgroup $\Gamma(\mathfrak{p})$ of $\mathrm{GL}(2, F)$.

The following table gives the dimensions of $\Gamma(\mathfrak{p})$–fixed vectors in the non-supercuspidal characters supported in the Borel, the Siegel parabolic, or the Klingen parabolic subgroup. The "$\mathrm{GSp}(4, \mathbb{F}_q)$" column indicates the finite representations producing the dimensions given.

Table 11: Dimensions of $\Gamma(\mathfrak{p})$–fixed vectors

		Representation	**Dimension**	$\mathrm{GSp}(4, \mathbb{F}_q)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$(q^2+1)(q+1)^2$	$\chi_1 \times \chi_2 \rtimes \sigma$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$q(q^2+1)(q+1)$	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$(q^2+1)(q+1)$	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$	$q(q^2+1)(q+1)$	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$(q^2+1)(q+1)$	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	q^4	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$	$q(q^2+q+1)$	$\sigma \mathrm{Ind}(\theta_9)_a + \sigma \mathrm{Ind}(\theta_{11})_a$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$q(q^2+q+1)$	$\sigma \mathrm{Ind}(\theta_9)_a + \sigma \mathrm{Ind}(\theta_{12})_a$
	d	$\sigma 1_{\mathrm{GSp}(4)}$	1	$\sigma 1_{\mathrm{GSp}(4)}$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$q^2(q^2+1)$	$\sigma \mathrm{Ind}(\theta_1)$
	b	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$q(q^2+1)$	$\sigma \mathrm{Ind}(\Phi_9)_a$
	c	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$q(q^2+1)$	$\sigma \mathrm{Ind}(\Phi_9)_b$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	q^2+1	$\sigma \mathrm{Ind}(\theta_3)$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	$q^4 + \frac{1}{2}q(q+1)^2$	$\sigma \mathrm{St}_{\mathrm{GSp}(4)} + \sigma \mathrm{Ind}(\theta_9)_a$
	b	$\tau(T, \nu^{-1/2}\sigma)$	$\frac{1}{2}q(q^2+1)$	$\sigma \mathrm{Ind}(\theta_{11})_a$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\frac{1}{2}q(q^2+1)$	$\sigma \mathrm{Ind}(\theta_{12})_a$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	$1 + \frac{1}{2}q(q+1)^2$	$\sigma 1_{\mathrm{GSp}(4)} + \sigma \mathrm{Ind}(\theta_9)_a$
VII		$\chi \rtimes \pi$ (irreducible)	q^4-1	$\chi \rtimes \pi$
VIII	a	$\tau(S, \pi)$	$q(q^2+1)(q-1)$	$\mathrm{Ind}(\omega_\pi \Phi_3)$
	b	$\tau(T, \pi)$	$(q^2+1)(q-1)$	$\mathrm{Ind}(\omega_\pi \Phi_1)$
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi)$	$q^2(q^2-1)$	$\mathrm{Ind}(\xi\omega_\pi\theta_5)$
	b	$L(\nu\xi, \nu^{-1/2}\pi)$	q^2-1	$\mathrm{Ind}(\xi\omega_\pi\theta_7)$
X		$\pi \rtimes \sigma$ (irreducible)	q^4-1	$\pi \rtimes \sigma$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$q(q^2+1)(q-1)$	$\sigma \mathrm{Ind}(\chi_7(n))_a$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$	$(q^2+1)(q-1)$	$\sigma \mathrm{Ind}(\chi_6(n))_a$

The dimensions in the table can be verified by comparing the decompositions of each type to the finite group decompositions of the corresponding type. Note that we may assume that $\sigma = 1$ since the space of fixed vectors doesn't change under twisting with characters trivial on $1 + \mathfrak{p}$. Indeed, let π be a representation of

$\mathrm{GSp}(4, F)$, σ a character of $F^\times$ which is trivial on $1 + \mathfrak{p}$, and v a $\Gamma(\mathfrak{p})$–fixed vector. Then, for all $g \in \Gamma(\mathfrak{p})$, we have

$$(\sigma\pi)(g)v = \sigma(\lambda(g))\pi(g)v = \sigma(\lambda(g))v = v.$$

The type VI representations require some additional information to compute dimensions using Table 10. The issue is where to place the common factor $\sigma\mathrm{Ind}(\theta_9)_a$. It is important to note that VId has a three–dimensional subspace of Iwahori subgroup–fixed vectors so the dimension of its space of $\Gamma(\mathfrak{p})$–fixed vectors is at least three–dimensional. Comparing with Table 10 completes the argument for the dimensions.

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